On a theorem of Fenchel on the total curvature of a closed curve.

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(Received April 15, 1957)

1. Let C be a space curve of length L:

$$C: \quad x=x(s)\,, \quad y=y(s)\,, \quad z=z(s)\,, \quad 0\leqq s\leqq L\,,$$

where s is the arc length of C, measured from a fixed point and $x(s)$, $x'(s)$, $x''(s)$ etc. are continuous periodic functions of s with a period L.

Let $k(s)=\lim\limits_{\Delta s\to 0}\dfrac{\Delta\theta}{\Delta s}=\sqrt{x''^2(s)+y''^2(s)+z''^2(s)}$ be the curvature of C at a point s, where $\Delta\theta$ is the angle between tangents of C at two points s and $s+\Delta s$. Then Fenchel[1] proved the following theorem.

Theorem 1. $$\int_0^L k(s)ds\geqq 2\pi\,,$$

where the equality holds, only when C is a plane convex curve.

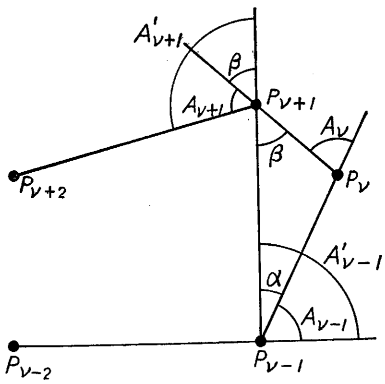
This can be deduced easily from the following Theorem 2.

2. First we shall prove a lemma.

Lemma. *Let Π_n be a space closed polygon of n sides and $P_\nu(\nu=1, 2, \cdots, n)$ be its vertices and A_ν $(0<A_\nu\leqq\pi)$ be the exterior angle of Π_n at P_ν and $\Theta_n=\sum\limits_{\nu=1}^{n}A_\nu$. Let Π_{n-1} be a closed polygon of $(n-1)$ sides, obtained from Π_n by taking off a vertex P_ν, then*

$$\Theta_n\geqq\Theta_{n-1}\,,$$

where the equality holds, only when $P_{\nu-2}$, $P_{\nu-1}$, P_ν, $P_{\nu+1}$, $P_{\nu+2}$ lie on a plane and $P_{\nu-2}P_{\nu-1}P_\nu P_{\nu+1}P_{\nu+2}$ is a convex polygon.



Proof. Let $A'_{\nu-1}$, $A'_{\nu+1}$ be the exterior angle of Π_{n-1} at $P_{\nu-1}$, $P_{\nu+1}$ respectively, then from the figure, we see that

$$A_{\nu-1}+A_\nu+A_{\nu+1}=(A_{\nu-1}+\alpha)+(A_{\nu+1}+\beta)\,.$$

Since $A_{\nu-1}+\alpha\geqq A'_{\nu-1}$, $A_{\nu+1}+\beta\geqq A'_{\nu+1}$, we have

$$A_{\nu-1}+A_\nu+A_{\nu+1}\geqq A'_{\nu-1}+A'_{\nu+1}\,,$$

1) W. Fenchel: Über Krümmung und Windung geschlossener Raumkurven, Math. Ann. **101** (1929).—S. Sasaki: On the total curvature of a closed curve. (in Japanese). Kikagaku-kenkyuhan. Hanpo No. **2** (1955).

so that

$$\Theta_n \geqq \Theta_{n-1} ,$$

where the equality holds, only when $P_{\nu-2}$, $P_{\nu-1}$, P_ν, $P_{\nu+1}$, $P_{\nu+2}$ lie on a plane and $P_{\nu-2}P_{\nu-1}P_\nu P_{\nu+1}P_{\nu+2}$ is a convex polygon.

By means of the lemma, we shall prove

Theorem 2. *Let Π_n be a space closed polygon of n sides and Θ_n be the sum of exterior angles at its vertices, then*

$$\Theta_n \geqq 2\pi ,$$

where the equality holds, only when Π_n is a plane convex polygon.

Proof. We take off a vertex from Π_n and form a closed polygon Π_{n-1} of $(n-1)$ sides and then we take off a vertex from Π_{n-1} and proceed similarly, then after a finite number of steps, we obtain a triangle, whose sum of exterior angles is 2π, so that by the lemma, $\Theta_n \geqq 2\pi$, where the equality holds, only when Π_n is a plane polygon and we see easily that Π_n is a convex polygon.

3. Now we shall prove Theorem 1. Let $P(s)=\{x(s),\ y(s),\ z(s)\}$ be the point of C, which corresponds to s and $\varDelta\psi$ be the angle between two vectors $\overrightarrow{P(s-\varDelta s)\ P(s)}$ and $\overrightarrow{P(s)\ P(s+\varDelta s)}$, then first we shall prove that

$$\lim_{\varDelta s\to 0}\frac{\varDelta\psi}{\varDelta s}=k(s) . \tag{1}$$

Now

$$x(s-\varDelta s)=x(s)-x'(s)\varDelta s+\frac{x''(s_1)}{2}(\varDelta s)^2 , \quad (s-\varDelta s<s_1<s)$$

$$x(s+\varDelta s)=x(s)+x'(s)\varDelta s+\frac{x''(s_2)}{2}(\varDelta s)^2 , \quad (s<s_2<s+\varDelta s) ,$$

etc., so that

$$(\sin\varDelta\psi)^2$$
$$=\frac{\sum[\{x(s+\varDelta s)-x(s)\}\{y(s)-y(s-\varDelta s)\}-\{x(s)-x(s-\varDelta s)\}\{y(s+\varDelta s)-y(s)\}]^2}{\sum\{x(s+\varDelta s)-x(s)\}^2\sum\{x(s)-x(s-\varDelta s)\}^2}$$
$$=\frac{\sum\left[\frac{y'(s)}{2}\{x''(s_1)+x''(s_2)\}-\frac{x'(s)}{2}\{y''(s_1')+y''(s_2')\}+O(\varDelta s)\right]^2}{\{\sum x'^2(s)+O(\varDelta s)\}\{\sum x'^2(s)+O(\varDelta s)\}}(\varDelta s)^2 .$$

Since $\sum x'^2(s)=1$, $\sum x'(s)x''(s)=0$, we have

$$\lim_{\varDelta s\to 0}\left(\frac{\varDelta\psi}{\varDelta s}\right)^2=\sum\{y'(s)x''(s)-x'(s)y''(s)\}^2=\sum x'^2(s)\sum x''^2(s)-\{\sum x'(s)x''(s)\}^2$$
$$=\sum x''^2(s)=k(s)^2 ,$$

hence

$$\lim_{\Delta s\to 0}\frac{\Delta\psi}{\Delta s}=k(s)\ .$$

Let s be measured from a fixed point P_0. We divide C into 2^n arcs of equal length $L/2^n$ and let $P_\nu\,(\nu=0, 1, 2, \cdots, 2^n-1)$ be the points of division, such that $\widehat{P_\nu\, P_{\nu+1}}=L/2^n$. Let Π_{2^n} be a closed polygon of 2^n sides, inscribed in C, whose vertices are $\{P_\nu\}$ and let Θ_{2^n} be the sum of exterior angles at its vertices, then $\Theta_{2^n}\geqq 2\pi$.
Since by (1),

$$\lim_n \Theta_{2^n}=\int_0^L k(s)ds\ ,$$

we have

$$\int_0^L k(s)ds\geqq 2\pi\ . \tag{2}$$

If C is not a plane curve, then if n is sufficiently large, Π_{2^n} is not a plane polygon, so that $\Theta_{2^n}>2\pi$. Since by the lemma, $\Theta_{2^{n+1}}\geqq\Theta_{2^n}$, we have

$$\int_0^L k(s)ds\geqq\Theta_{2^n}>2\pi\ . \tag{3}$$

Hence if the equality in (2) holds, then C is a plane curve and we see easily that C is a convex curve.

4. Fenchel deduced Theorem 1 from the following

Theorem 3. *Let C be a rectifiable closed space curve of length L on a unit sphere S, which is not a plane curve, such that the center O of S is an inner point of the smallest convex set, which contains C, then*

$$L>2\pi\ .$$

This can be proved simply by means of Theorem 2 as follows.

Proof. We may assume that O is the origin of coordinates. Let s be the arc length of C, measured from a fixed point and $\mathfrak{x}(s)$ be the point of C, which corresponds to s. We divide C into N arcs of equal length L/N and let $\mathfrak{x}_\nu\,(\nu=1, 2, \cdots, N)$ be the points of division, such that the arc length $\widehat{\mathfrak{x}_\nu\, \mathfrak{x}_{\nu+1}}=L/N$. By the hypothesis on O, if N is sufficiently large, O is an inner point of the smallest convex set, which contains N points $\{\mathfrak{x}_\nu\}$, so that there exists $0\leqq\lambda_\nu<1$, such that

$$\lambda_1\mathfrak{x}_1+\cdots+\lambda_N\mathfrak{x}_N=0\ ,\qquad \lambda_1+\cdots+\lambda_N=1\ .$$

Some of λ_ν may be zero and let λ_{ν_k} $(k=1, 2, \cdots, n)$ be such ones, which are not zero, so that

$$\lambda_{\nu_1}\mathfrak{x}_{\nu_1}+\cdots+\lambda_{\nu_n}\mathfrak{x}_{\nu_n}=0\ ,\ \lambda_{\nu_1}+\cdots+\lambda_{\nu_n}=1\ (1\leqq\nu_1<\nu_2<\cdots<\nu_n\leqq N)\ . \quad (1)$$

If $n=2$, then $\mathfrak{x}_{\nu_1}$, $\mathfrak{x}_{\nu_2}$ lie diametrically, so that an arc of C, which connects $\mathfrak{x}_{\nu_1}$, $\mathfrak{x}_{\nu_2}$ has a length $\geqq\pi$. Since there are two such arcs of C, we have $L\geqq 2\pi$. If $L=2\pi$, then we see easily that C coincides with the great circle of S, through $\mathfrak{x}_{\nu_1}$, $\mathfrak{x}_{\nu_2}$ which contradicts the hypothesis, hence

$$L>2\pi\ . \quad (2)$$

Hence we assume that $n\geqq 3$. Then by (1), n vectors $\lambda_{\nu_k}\mathfrak{x}_{\nu_k}$ $(k=1, 2, \cdots, n)$ can be considered as n sides of a closed polygon Π_n, which is defined as follows. Let P_1 be the end point of $\lambda_{\nu_1}\mathfrak{x}_{\nu_1}$ and we draw a vector $\lambda_{\nu_2}\mathfrak{x}_{\nu_2}$ with P_1 as its initial point. Let P_2 be the end point of $\lambda_{\nu_2}\mathfrak{x}_{\nu_2}$ and we draw a vector $\lambda_{\nu_3}\mathfrak{x}_{\nu_3}$ with P_2 as its initial point and son on, then we obtain Π_n. P_k $(k=1, 2, \cdots, n)$ are vertices of Π_n and the exterior angle A_k of Π_n at P_k is the angle between $\mathfrak{x}_{\nu_k}$ and $\mathfrak{x}_{\nu_{k+1}}$, which is the arc length of the great circle of S, bounded by $\mathfrak{x}_{\nu_k}$ and $\mathfrak{x}_{\nu_{k+1}}$, so that the arc length of C, bounded by $\mathfrak{x}_{\nu_k}$ and $\mathfrak{x}_{k+1}$ is $\geqq A_k$, hence

$$L\geqq\sum_{k=1}^{n} A_k=\Theta_n\geqq 2\pi\ . \quad (3)$$

If the equality holds, then n points $\{\mathfrak{x}_{\nu_k}\}$ lie on a plane and C coincides with the great circle of S through $\{\mathfrak{x}_{\nu_k}\}$, which contradicts the hypothesis. Hence

$$L>2\pi\ . \quad (4)$$

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